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Is a fully-developed and non-isothermal flow possible in a vertical pipe?

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Abstract—The results of a linear-stability analysis of the fully-developed flow in a heated vertical pipe are presented. They confirm the experimental observations that flow in a heated vertical pipe is supercritically unstable. The bifurcated new equilibrium laminar flow is likely to be a double spiral flow. Mixing induced by this spiral flow can cause a substantial increase in the heat-transfer rate and even delay transition to turbulence, as has been observed experimentally.

1. INTRODUCTION

HEAT TRANSFER in a circular pipe is a fundamental convection problem. Unfortunately, our understanding of the physics of fluid flows and the associated heat-transfer mechanisms is, in fact, incomplete and in many respects, erroneous. For example, most textbooks still describe a fully-developed flow in a heated pipe as a *parallel* flow, similar to an isothermal flow. In this case, the velocity and temperature distributions are functions of the radial coordinate only. Consequently, many design methods for thermal systems, which contain various kinds of straight tubes, have been developed solely on this simple, but, as will be shown below, erroneous assumption. A more surprising fact is that many research papers appearing in archival journals contain the same mistake. In this paper we will demonstrate that a flow in a heated vertical pipe is highly unstable; therefore, a parallel, fully-developed flow can hardly be expected in any engineering system.

An isothermal flow in a straight pipe can be classified into three regions: entry, developing and fully-developed flows [1]. An entry flow consists of a thin viscous layer near the pipe wall and an almost inviscid core flow. After the viscous layer fills the entire cross section of the pipe and before the flow becomes independent of the axial distance, the flow is defined as developing. The typical characteristics of a fully-developed flow are: (1) the axial pressure gradient is constant and the pressure is uniform across the pipe, (2) the radial velocity component is zero and the flow is parallel and independent of the axial distance. In the other two regions, the flow cannot be analyzed with these simplifications. It is known that the entry flow occupies about 25% of the entry length; thus, the flow in a thermal system is likely to be a developing one.

Flows in a *heated* straight pipe behave quite differently from isothermal flows. For a *horizontal* pipe, secondary flows are induced by the density stratification [2–9]. Consequently, the criterion for flow transition and the heat transfer mechanism are completely altered by the secondary motion.

An extensive list of references for flows in a heated vertical pipe can be found in ref. [10]. With few exceptions [11–14], most studies have adopted the assumption of laminar and/or parallel flow. Scheele and Hanratty [11] obtained an analytical solution for fully-developed flow in a heated vertical tube, and observed experimentally that the flow is stable in the entry region but highly unstable in the fully-developed region. They speculated that the flow goes through transition at rather low Reynolds number, but the flow, unlike other turbulent flows, consists of large-scale, regular and periodic motions. Similar flow patterns have been observed and are called *nonlaminar* by Kemeny and Somers [12] to distinguish them from turbulent patterns. The striking fact is that flows can become nonlaminar at Reynolds numbers as low as 30. They also found that the non-laminar heat transfer rate can be 30% larger than those in laminar flow. Unfortunately, they ignored these 'unusual' non-laminar data in deriving their Nusselt number correlations. Similar phenomenon and reversed-flow transition due to the mixing introduced by large-scale periodic, secondary flows have been observed by Steiner [13]. Analytical considerations [14] suggest that the developing lengths corresponding to the entry and the developing flows in a heated vertical tube are shorter than those in an unheated tube. Steiner's data have been successfully correlated in terms of the parameters suggested in ref. [14]. In this paper, the stability of a fully-developed upward flow in a heated vertical pipe (or a downward flow in a cooled tube) is studied. The results suggest that a heated flow in a vertical tube is extremely unstable (see Fig. 5) and that the instability is supercritical. Except for a narrow range of Reynolds numbers, the disturbed flow has a double-spiral structure. This agrees with the observation of refs. [11–13] that the instability is super-

NOMENCLATURE

a	radius of the pipe	T	temperature
c	complex wave speed, $c_r + i c_i$	u	radial velocity
D	operator defined in equations (2b) and (9a)	v	azimuthal velocity
f, g	streamfunctions defined in equations (10a)–(10c)	w, W	axial velocity
$\bar{g}$	gravitational acceleration	z	axial coordinate.
Gr	Grashof number, $\beta\bar{g}\tau a^4/\nu^2$		
J	operator defined in equation (2a)	Greek symbols	
L	operator defined in equation (9b)	α	wave number in equation (7)
n	azimuthal modes	$\bar{\alpha}$	thermal diffusivity
p, P	pressure	β	thermal expansion coefficient
r	radial coordinate	ν	kinematic viscosity
Ra	Rayleigh number, $\beta\bar{g}\tau a^4/\nu\bar{\alpha}$	ρ	density
Re	Reynolds number	θ	dimensionless temperature in equation (3a)
t	time	τ	axial temperature gradient along the pipe wall.

critical and a new, equilibrium, laminar and non-parallel flow exists after flow bifurcation. Moreover, it seems that some of the unstable flows observed by Scheele and Hanratty occurred in the developing region instead of in the fully-developed region, as speculated by them.

A brief review of the stability of isothermal Hagen–Poiseuille flow is relevant to our study. The stability of this flow has been of continuing interest since Reynolds' classic experiments. Theoretical and experimental studies [15–24] have shown that Hagen–Poiseuille flow is stable for infinitesimal disturbance. Non-linear analyses [25–31] have, however, indicated that a subcritical instability is possible for finite-amplitude disturbances. A more relevant stability problem is the spiral Poiseuille flow between rotating cylinders [32–34]. It has been found that the axial flow can enhance the Taylor instability induced by the rotating cylinders, and higher azimuthal modes become unstable when the axial Reynolds number increases. This differs from the flow instability in a heated vertical tube in which $n = 1$ (see equation (7)) is the dominant azimuthal mode. The two problems, however, share many similarities: (1) their instabilities are supercritical and new equilibrium laminar flows exist after flow bifurcations; and (2) the fully-developed flow is less stable than the corresponding developing flow.

A linear stability analysis is performed in the following section for an upward fully-developed flow in a heated vertical pipe (which is identical to the downward flow in a cooled vertical pipe). One of the reasons for studying the stability of the fully-developed flow is that stability problems for a parallel flow are relatively simple; in contrast, our knowledge of instability analysis for a non-parallel flow is incomplete today [35]. Upstream influences induced by low-frequency unstable waves must be included in the analysis in order to have a complete theory for non-parallel flows. Three-dimensional computations are probably required to consider the full elliptical nature of the stability equations and their interaction with the mean-flow equations. Further research is needed before a satisfactory theorem becomes available. Our results, presented in Section 4, for a fully-developed flow can, however, elucidate some of the physics of this complex problem, and the conclusion in Section 5, we believe, can certainly be extended to the more important case of a developing flow.

2. FORMULATION

For an incompressible flow, the dimensionless Navier–Stokes and energy equations, in terms of cylindrical polar coordinates (r, ψ, z), are

$$\frac{\partial\hat{u}}{\partial r} + \frac{\hat{u}}{r} + \frac{1}{r}\frac{\partial\hat{v}}{\partial\psi} + \frac{\partial\hat{w}}{\partial z} = 0 \tag{1a}$$

$$\frac{\partial\hat{u}}{\partial t} + J\hat{u} - \frac{\hat{v}^2}{r} = -\frac{\partial\hat{p}}{\partial r} + \frac{1}{Re}\left(D_3^2\hat{u} - \frac{\hat{u}}{r^2} - \frac{2}{r^2}\frac{\partial\hat{v}}{\partial\psi}\right) \tag{1b}$$

$$\frac{\partial\hat{v}}{\partial t} + J\hat{v} - \frac{\hat{u}\hat{v}}{r} = -\frac{1}{r}\frac{\partial\hat{p}}{\partial\psi} + \frac{1}{Re}\left(D_3^2\hat{v} - \frac{\hat{v}}{r^2} + \frac{2}{r^2}\frac{\partial\hat{u}}{\partial\psi}\right) \tag{1c}$$

$$\frac{\partial\hat{w}}{\partial t} + J\hat{w} = -\frac{\partial\hat{p}}{\partial z} - \frac{Ra}{Re}\hat{\theta} + \frac{1}{Re}D_3^2\hat{w} \tag{1d}$$

and

$$\frac{\partial \hat{\theta}}{\partial t} + J\hat{\theta} - \frac{\hat{w}}{Re\,Pr} = \frac{1}{Re\,Pr} D_3^2 \hat{\theta} \tag{1e}$$

where

$$J = \hat{u}\frac{\partial}{\partial r} + \frac{\hat{v}}{r}\frac{\partial}{\partial \psi} + \hat{w}\frac{\partial}{\partial z} \tag{2a}$$

and

$$D_3^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \psi^2} + \frac{\partial^2}{\partial z^2}. \tag{2b}$$

The coordinates are nondimensionalized by the radius of the cylinder, a, the velocities by the mean axial velocity, $\bar{w}$, the pressure by $\rho\bar{w}^2$, and the time by $a/\bar{w}$. The temperature distribution along the pipe wall is assumed to be linear with a constant axial gradient, τ; thus

$$T = T_w - (\tau a\,Re\,Pr)\hat{\theta} \tag{3}$$
$$T_w = T_0 + \tau a z$$

where T_0 is the upstream reference temperature. The parameters are Reynolds number $Re = \bar{w}\,a/v$, Prandtl number $Pr = \bar{\alpha}/v$, and Rayleigh number $Ra = \beta\bar{g}\tau a^4/v\bar{\alpha}$, where $\bar{g}$ is the gravitational acceleration, v the kinematic viscosity, β the thermal expansion coefficient, and $\bar{\alpha}$ the coefficient of thermal diffusivity. The surface temperature is zero because of equation (3).

2.1. *Mean flow*

Since we wish to study the stability of an upward, fully-developed flow in a heated circular pipe, the dependent variables in equations (1a)–(1e) are split into two parts: mean flow and disturbance. If the classical assumption of fully-developed flow is applied to equations (1a)–(1e), it can be shown that the mean flow is independent of the axial and azimuthal coordinates. The governing equations of the mean flow are, with mean-flow quantities denoted by capital letters

$$D^2 W = Re\frac{\mathrm{d}P}{\mathrm{d}z} + Ra\,\Theta \tag{4a}$$

$$D^2\Theta = -W; \tag{4b}$$

the axial pressure gradient can be determined by the requirement of global mass conservation

$$\int_0^1 rW\,\mathrm{d}r = 1/2. \tag{4c}$$

The solutions of equations (4a)–(4c) satisfying the no-slip conditions on the pipe wall ($r = 1$), and bounded along the pipe center can be found from ref. [36]. They are

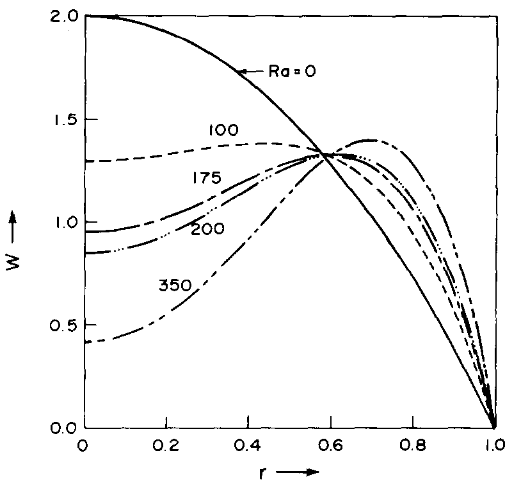

FIG. 1. Mean-flow velocity profile.

$$W = c_1\,\mathrm{ber}(r\,Ra^{1/4}) + c_2\,\mathrm{bei}(r\,Ra^{1/4}) \tag{5a}$$

$$-Ra^{1/2}\,\Theta = c_1[\mathrm{bei}(r\,Ra^{1/4}) - \mathrm{bei}(Ra^{1/4})] - c_2[\mathrm{ber}(r\,Ra^{1/4}) - \mathrm{ber}(Ra^{1/4})] \tag{5b}$$

with

$$c_1 = 0.5Ra^{1/4}\,\mathrm{bei}(Ra^{1/4})/[\mathrm{ber}'(Ra^{1/4})\,\mathrm{ber}(Ra^{1/4}) + \mathrm{ber}'(Ra^{1/4})\,\mathrm{bei}(Ra^{1/4})] \tag{5c}$$

and

$$c_2 = -c_1\,\mathrm{ber}(Ra^{1/4})/\mathrm{bei}(Ra^{1/4}) \tag{5d}$$

where

$$\mathrm{ber}(x) + \mathrm{i}\,\mathrm{bei}(x) = J_0\left(\frac{-1+\mathrm{i}}{\sqrt{2}}x\right) \tag{5e}$$

and J_0 is the first kind Bessel function of order zero.

The mean-flow velocity profiles for $Ra = 0$, 100, 175, 200 and 350 are plotted in Fig. 1. Near the pipe wall, the fluid flows faster due to heating. Consequently, the fluid near the center of the pipe is slowed in order to satisfy mass conservation. It is clear that the velocity profile contains an inflection point and is likely to be unstable. The distributions of mean temperature are given in Fig. 2 for $Ra = 0$, 100 and 175, respectively.

Due to the different nondimensionalization of the equations in this paper and those of Hanratty *et al.* [36], the relation between the governing parameters is needed in order to compare results: $(Gr/Re)_{\mathrm{HRK}}$ in ref. [36] equals $Ra\cdot\Theta(0)$ in this paper. The centerline temperature $\Theta(0)$ is plotted in Fig. 3 as a function of Ra and the relation between $(Gr/Re)_{\mathrm{HRK}}$ and Ra is provided in Fig. 4.

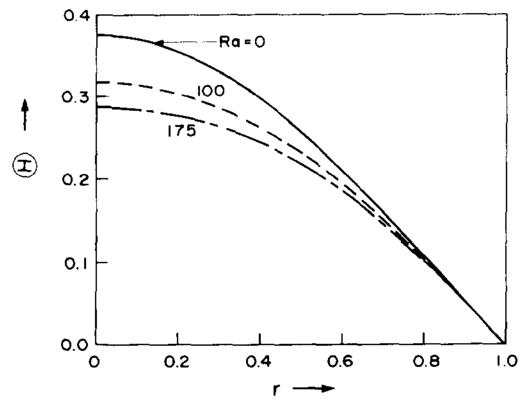

FIG. 2. Mean temperature distribution.

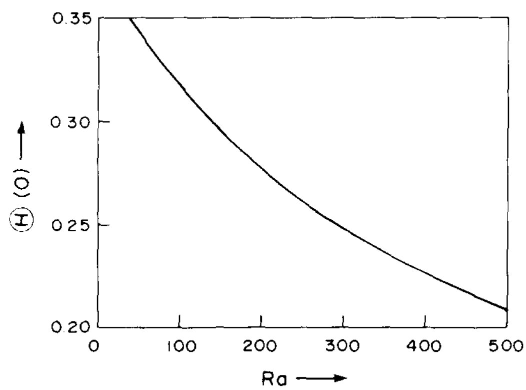

FIG. 3. Centerline temperature.

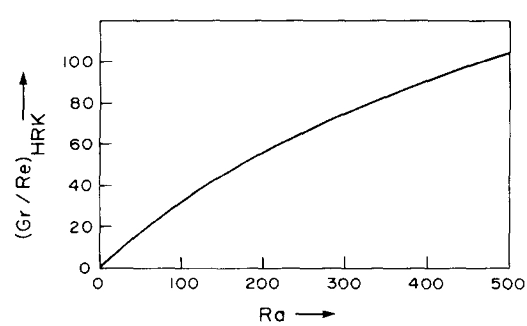

FIG. 4. Relation of *Gr/Re* in ref. [36] with *Ra* of this paper.

2.2. *Disturbance*

The linear equations governing the disturbances can be obtained by subtracting the mean-flow equations, equations (4a)–(4c), from the full equations, equations (1a)–(1e), and neglecting the small non-linear terms. They are

$$\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r} + \frac{1}{r}\frac{\partial \bar{v}}{\partial \psi} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{6a}$$

$$\frac{\partial \bar{u}}{\partial t} + W\frac{\partial \bar{u}}{\partial z} = -\frac{\partial \bar{p}}{\partial r} + \frac{1}{Re}\left(D_3^2 \bar{u} - \frac{\bar{u}}{r^2} - \frac{2}{r^2}\frac{\partial \bar{v}}{\partial \psi}\right) \tag{6b}$$

$$\frac{\partial \bar{v}}{\partial t} + W\frac{\partial \bar{v}}{\partial z} = -\frac{1}{r}\frac{\partial \bar{p}}{\partial \psi} + \frac{1}{Re}\left(D_3^2 \bar{v} - \frac{\bar{v}}{r^2} + \frac{2}{r^2}\frac{\partial \bar{u}}{\partial \psi}\right) \tag{6c}$$

$$\frac{\partial \bar{w}}{\partial t} + W\frac{\partial \bar{w}}{\partial z} + W'\bar{u} = -\varepsilon\bar{\theta} - \frac{\partial \bar{p}}{\partial z} + \frac{1}{Re}D_3^2 \bar{w} \tag{6d}$$

$$\frac{\partial \bar{\theta}}{\partial t} + W\frac{\partial \bar{\theta}}{\partial z} + u\Theta' - \frac{\bar{w}}{Pr\,Re} = \frac{1}{Pr\,Re}D_3^2 \bar{\theta}. \tag{6e}$$

It is easy to check that equations (4) and (6) are invariant under the transformation $(W, \bar{w}, \Theta, \bar{\theta}, z) \to (-W, -\bar{w}, -\Theta, -\bar{\theta}, -z)$. Thus, the mean flow as well as the stability characteristics of a heated upward flow are identical to those of a cooled downward flow.

Equations (6a)–(6e) can be reduced to a set of ordinary differential equations if the disturbance quantities are represented in a normal-mode form, such as

$$\bar{\phi} = \phi(r)\,e^{i[\alpha(z-ct)+n\psi]} \tag{7}$$

where α is the wave number, n the integer azimuthal wave number, and c the complex wave speed. Substitution of equation (7) into equations (6a)–(6e) yields

$$\left(D + \frac{1}{r}\right)u + \frac{in}{r}v + i\alpha w = 0 \tag{8a}$$

$$i\alpha(W-c)u + Dp - \frac{1}{Re}\left(Lu - \frac{i2n}{r^2}v\right) = 0 \tag{8b}$$

$$i\alpha(W-c)v + \frac{in}{r}p - \frac{1}{Re}\left(Lv + \frac{i2n}{r^2}u\right) = 0 \tag{8c}$$

$$i\alpha(W-c)w + u\,DW + \varepsilon\theta + i\alpha p - \frac{1}{Re}\left(Lw + \frac{w}{r^2}\right) = 0 \tag{8d}$$

and

$$i\alpha(W-c)\theta + u\,D\Theta - \frac{1}{Pr\,Re}\left(w + L\theta + \frac{\theta}{r^2}\right) = 0 \tag{8e}$$

where

$$D = \frac{d}{dr} \tag{9a}$$

and

$$L = D\left(D + \frac{1}{r}\right) - (\alpha^2 + n^2)/r^2. \tag{9b}$$

Equations (8a)–(8e) can be reduced to three equations by eliminating p among equations (8b)–(8d) and by introducing two stream functions, f and g, which are defined by

$$w = f'(r) + f/r \tag{10a}$$

$$v = g'(r) \tag{10b}$$

$$u = -\mathrm{i}\alpha f - \frac{\mathrm{i}\, n}{r} g \tag{10c}$$

where the prime denotes differentiation with respect to r. Equations (8a)–(8e), then, become

$$\begin{aligned}
&f^{(4)} + 2f'''/r - [(n^3 + 3)/r + 2\alpha^2 + \mathrm{i}\,\alpha\, Re\, W]f'' \\
&\quad + [(n^2 + 3)/r^2 - 2\alpha^2 - \mathrm{i}\,\alpha\, Re\, W]f'/r \\
&\quad + [(3n^2 - 3)/r^4 + \alpha^2(n^2 + 2)/r^2 + \alpha^4 \\
&\quad + \mathrm{i}\,\alpha\, Re(D^2 - D/r + 1/r^2 + \alpha^2)W]f \\
&\quad - n\alpha g''/r - n(\alpha/r - \mathrm{i}\, Re\, W')g'/r + n[\alpha n^2/r^2 + \alpha^3 \\
&\quad + \mathrm{i}\, Re(D^2 - D/r + \alpha^2)W]g/r - Ra\,\theta' \\
&= -\mathrm{i}\,\alpha\, Re\, c[(D^2 + D/r + 1/r^2 + \alpha^2 + n\,\alpha\, g/r] \qquad (11a)
\end{aligned}$$

$$\begin{aligned}
&g^{(4)} + 2g'''/r - [(2n^2 + 1)/r^2 + \alpha^2 + \mathrm{i}\,\alpha\, Re\, W]g'' \\
&\quad + [(2n^2 + 1)/r^3 - \alpha^2/r \\
&\quad - \mathrm{i}\,\alpha\, Re(D + 1/r)W]g' + n^2[(n^2 - 4)/r^4 \\
&\quad + (\alpha^2 + \mathrm{i}\,\alpha\, Re\, W)/r^2]g - n\,\alpha f''/r \\
&\quad + n\alpha f'/r^2 + n\alpha[(n^2 - 1)/r^2 \\
&\quad + (\alpha^2 + \mathrm{i}\,\alpha\, Re\, W)]f/r \\
&= \mathrm{i}\,\alpha\, Re\, c[n\,\alpha f/r - (D^2 + D/r - n^2/r)g] \qquad (11b)
\end{aligned}$$

$$\begin{aligned}
&\theta'' + \theta'/r - (n^2/r^2 + \alpha^2 + \mathrm{i}\,\alpha\, Re\, Pr\, W)\theta \\
&\quad + (D + 1/r)f + Re\, Pr\, \Theta'(\mathrm{i}\,\alpha f + \mathrm{i}\, n g/r) \\
&\qquad = -\mathrm{i}\,\alpha\, Re\, Pr\, c\,\theta. \qquad (11c)
\end{aligned}$$

The required boundary conditions on the pipe wall reflect the no-slip and no-penetration conditions, and the zero surface temperature. They are

$$f' + f = g' = \alpha f + ng = \theta = 0 \quad \text{at}\, r = 1. \tag{12a}$$

One more required condition can be obtained by eliminating p between equations (8c) and (8d); the result is

$$\alpha(D^3 + D^2)g - n(D^3 + 2D^2 - D + 1)f = 0. \tag{12b}$$

The conditions at $r = 0$ are more intriguing and purely kinematic [37]. They must be considered separately for $|n| = 0$, 1 and $|n| \geqslant 2$.

(i) $n = 0$: the disturbance is rotationally symmetric and two kinds of normal modes exist (meridional and torsional modes). The required conditions are $u = v = w' = \theta' = 0$, or

$$f = f'' = g' = \theta' = 0 \qquad \text{at}\, r = 0 \tag{13a}$$

where g is set to zero at $r = 0$ in order to uniquely determine it.

(ii) $|n| = 1$: the disturbances are two plane symmetric spiral flows. In this case, the radial and the azimuthal velocities are related; therefore, the conditions are $u + \mathrm{i}\, v = w = \theta = 0$, or

$$f = f' = g = g'' = \theta = 0 \quad \text{at}\, r = 0. \tag{13b}$$

(iii) $|n| \geqslant 2$: the normal mode is $2n$ spiral flows. The conditions are $u = v = w = \theta = 0$, or

$$f = f' = g = g' = \theta = 0 \quad \text{at}\, r = 0. \tag{13c}$$

Equations (11a)–(11c) and boundary conditions (12) and (13) form an eigenvalue problem. The instability boundary of the flow in the (Re, Ra) space is determined by the fact that the imaginary part of the complex wave speed c_i equals zero. This also forms a minimax problem with $c_i = 0$ when Re and Ra are local minima for various wave numbers α and n. A large number of computations is required in order to determine the flow instability boundary.

3. METHOD OF SOLUTION

A pseudo-spectral Chebyshev method [38] is used to discretize equations (11a)–(11c) and to incorporate boundary conditions (12) and (13). The collocation points are selected to be the extrema of Nth order Chebyshev polynomials in order to minimize the truncation error. The results have been compared with those obtained for grids of equal distance. The difference between the two grid systems is very slight. The eigenvalues of the matrix produced by the pseudo-spectral method are determined by a complex QR algorithm [39]. The method is very stable and the required computer time is moderate. The advantages of the method are that the numerical results provide a global instability map and no guesses are required. This feature is critical to a *new* flow instability problem, such as the one studied in this paper, for which the parameter domain for instability is not known *a priori*.

Computations have been carried out to check the accuracy of the numerical method by comparison with the previous published results for an isothermal tube flow [23]. Our results with 51 terms agree with those of ref. [23] to the third decimal point. It is worthwhile to note that additional unstable modes may occur due to the existence of the energy equation. The energy equation is removed from our computation when the results are compared with ref. [23]. Further comparison of our numerical results with various N shows that the results for 51 terms also

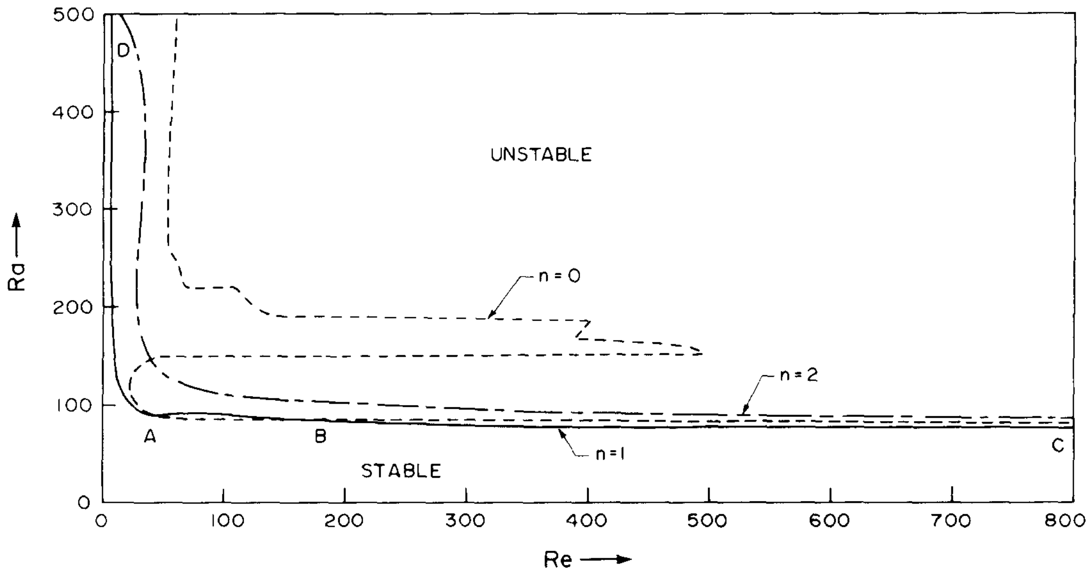

FIG. 5. Flow instability boundary in (Re, Ra) plane.

agrees with those for 71 or 91 terms to the third decimal point. The results discussed below are produced by Chebyshev polynomials with 51 terms.

4. RESULTS AND DISCUSSION

The flow instability boundaries for $n = 0$, 1 and 2 are plotted in Fig. 5. It is clear that $n = 1$ is the most unstable mode except in a very small region (denoted by AB) where $n = 0$ is the most unstable mode. This explains why the observed unstable flow pattern [11, 36] is a double spiral flow. It is known that pipe flow is stable for infinitesimal disturbances. On the other hand a heated tube flow is highly unstable. The boundary DAB in Fig. 5 indicates that a flow can become unstable for $Ra > 75$ and $Re > 40$, which means that a very slow flow at a rather mild heating condition is already unstable. The boundary ABC is due to the shear instability of a mean-flow velocity profile modified slightly by heating. The boundary DA is due to thermal instability. The density of the fluid in a heated tube without motion is stably stratified in the vertical direction. A slow flow can carry denser fluid upward into the region of lighter fluid. This is why the flow becomes unstable at extremely slow speeds at a high heating condition. The boundary between the shear instability and the thermal instability is not clear for our problem. This differs from the instability of a thermally stratified horizontal plane Poiseuille flow [40] in which a clear boundary exists between the shear and thermal instabilities in the parameter plane (Ra, Re). We will further probe this point later on by investigating the distribution of unstable modes in the (α, Ra) parameter plane for a fixed Re.

The instability boundary ABC was observed by Scheele and Hanratty [11]. Within the range of Re

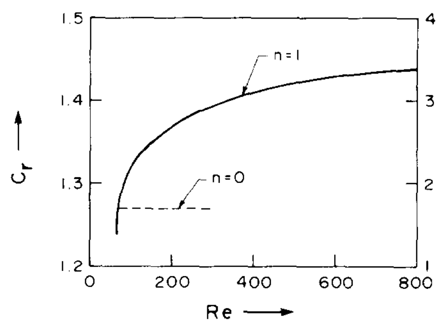

FIG. 6. Speed of shear instability wave.

in Fig. 5, the experimental results indicate that the flow is unstable when $Ra > 140$ which is larger than the value, $Ra > 80$, predicted by our linear stability analysis. The discrepancy can be attributed to two facts. First, the instability cannot be observed until its amplitude reaches an observable magnitude. Scheele and Hanratty's experiment was performed more than a quarter of a century ago. None of the modern electronic devices were available at that time to help them to accurately detect the *onset* of the flow instability. Second, the flow instability detected by them is very likely in the region of a developing flow whose mean flow profile can differ substantially from the fully-developed velocity profile used in this analysis. Since the developing flow is more stable than a fully-developed flow, the observed critical Rayleigh number is naturally larger than the prediction for a fully-developed flow.

The wave speed (the real part of c, c_r) along ABC of Fig. 5 is plotted as a function of Re in Fig. 6. The wave speed increases with Re. Comparing the wave speed with the mean-flow velocity shows that two critical layers exist. This is probably a typical property of a thermally disturbed flow. The wave number is

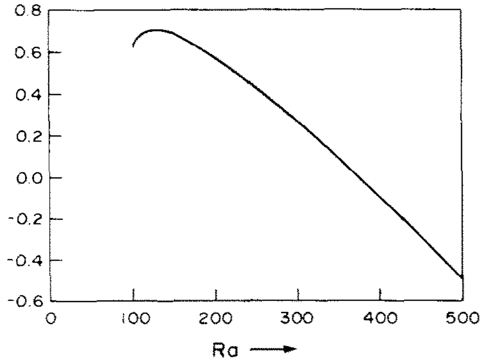

FIG. 7. Speed of thermal instability wave.

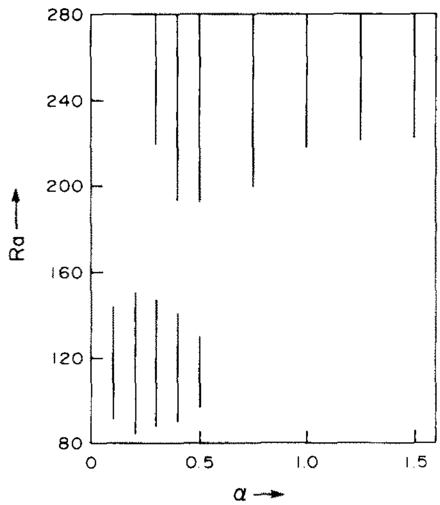

FIG. 8. Flow instability map in (α, Ra) plane for $n = 0$ and $Re = 300$.

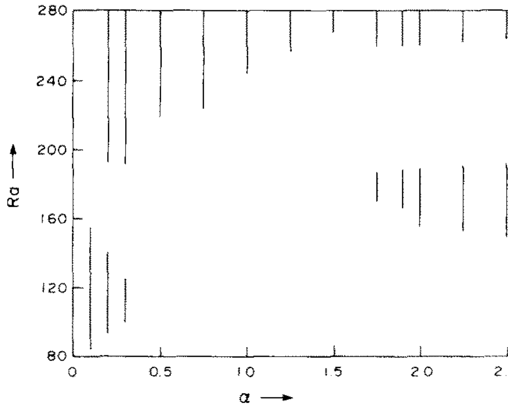

FIG. 9. Flow instability map in (α, Ra) plane for $n = 0$ and $Re = 600$.

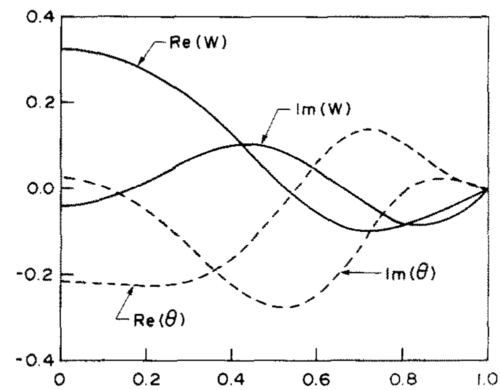

FIG. 10. Eigenfunctions of w and θ for $n = 0$, $\alpha = 0.1$, $Re = 600$, $Ra = 84.06$.

insensitive to the variation of Re. For $Re \geqslant 100$, $\alpha \simeq 3$ and for $Re = 70$, α drops to 2. This indicates that the wavelength is about the diameter of the pipe for a shear instability.

For a thermal instability (along AD of Fig. 5), the wave speed is plotted as a function of Ra in Fig. 7. The wave speed decreases when heating increases. A maximum occurrence at $Re \simeq 130$ is likely in the region where the flow instability shifts from the thermally triggered one to the shear instability. The wave number along AD is about 1.75 and is not sensitive to the variation of Ra, either. This means that the wavelength of thermal instability is always 1.7 times longer than that of the shear instability wave.

The irregular shape of the instability boundary for $n = 0$ in Fig. 5 induces our interest to probe the detailed distribution of unstable flow modes. For $Re = 300$, a stable flow region exists between $150 \leqslant Ra < 190$. A flow instability map in the (α, Ra) plane is given in Fig. 8. The solid lines mark the unstable region for discrete wave numbers. It clearly shows that two unstable flow regions exist. The one for small Ra is closed and is due to the shear instability; the one for large Ra is open and is due to the thermal instability. In between these two regions, the flow is stable. This stable flow region disappears when Re increases beyond $Re = 500$. A typical case of $Re = 600$ is selected to study the reason why this stable flow region disappears at large Re. In Fig. 9, a third unstable region appears which corresponds to the disappearance of the middle stable flow region when $Re > 500$. In this region, the flow becomes unstable due to the interaction of thermal and shear instabilities.

For $n = 0$ and $Re = 600$, the eigenfunctions for w and θ are plotted in Figs. 10–12 for the three least stable modes. They seem to suggest that there are two parallel travelling waves of 'donut' shape that exist for $Ra = 84.06$ and $\alpha = 0.1$. Similar eigenfunctions are observed for the thermally unstable region where $Ra = 191.91$ and $\alpha = 0.3$. In the interaction region, $Ra = 150.82$ and $\alpha = 2.5$, the eigenfunction is rather flat in most regions of the cross section of the pipe, and drastic changes occur only near the pipe wall. A comparison of eigenfunctions for various cases indicate that the eigenfunctions are insensitive to changes in α and Re and are functions of Ra only.

Interesting information can be revealed by follow-

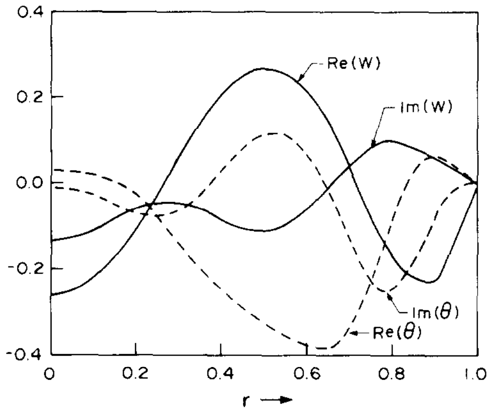

FIG. 11. Eigenfunctions of w and θ for $n = 0$, $\alpha = 0.3$, $Re = 600$, $Ra = 191.91$.

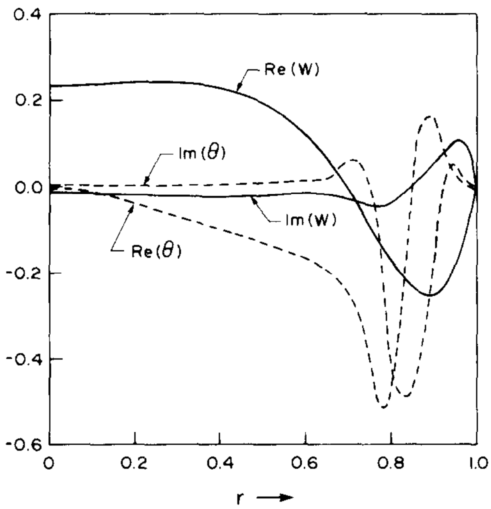

FIG. 12. Eigenfunctions of w and θ for $n = 0$, $\alpha = 2.5$, $Re = 600$, $Ra = 150.82$.

ing the evolution of the flow development with increasing Ra. The wave speed, c_r, and the wave amplification factor, c_i, for $n = 0$, $\alpha = 1$ and $Re = 100$ are plotted in Fig. 13. Modes 1 and 2 are thermal modes, 3 is a meridional mode, and 4 is a torsional mode. The thermal mode represents the instability induced by the onset of a temperature instability, the meridional mode is due to the instability of the axial and radial velocities, and the torsional mode is due to the azimuthal velocity. It is interesting to note that the flow instability, for this particular case, is first triggered by the unstable temperature distribution in $90 \leq Ra \leq 144$. The eigenfunctions show that, once the temperature distribution becomes unstable, w and u are also unstable, but v remains zero. The disturbance is axisymmetric. For $Ra > 210$, the most unstable mode is meridional; the torsional mode is always stable.

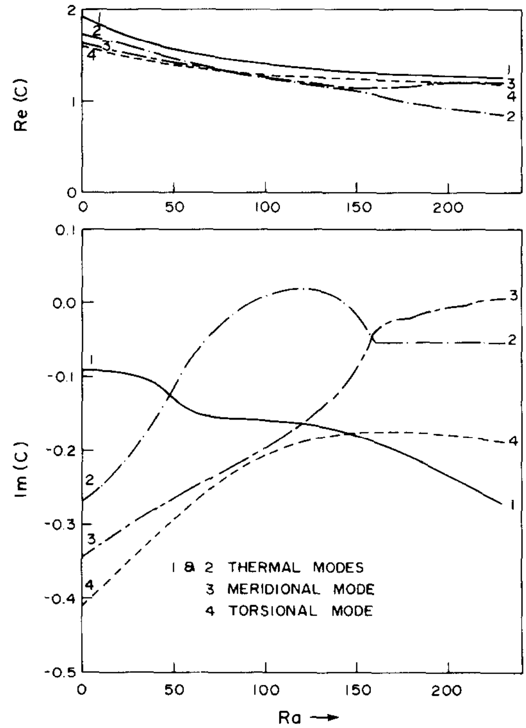

FIG. 13. Evolution of c_r (wave speed) and c_i (amplification factor) with respect to Ra.

5. CONCLUSION

The linear stability analysis indicates that an upward flow in a heated vertical pipe (or a downward flow in a cooled pipe) is unstable. The predicted critical values of Ra and Re agree reasonably well with experimental observation. This suggests that idealized fully-developed, forced-convection flows and/or mixed-convection flows can only exist with minimum heating (or cooling). The heat-transfer data, previously analytically predicted or experimentally correlated, without considering flow instability, which are widely cited in textbooks, should be used with caution. The same conclusion can be extended to developing flows as well. We also believe that the instability associated with the double spiral flow in a heated pipe is supercritical. A new equilibrium laminar state can exist within the unstable region predicted by the linear stability analysis. Since the wavelength is of the order of the pipe diameter this verifies the experimental observation that the disturbance size is large. The mixing introduced by the spiral flow can certainly delay the mechanism of flow transition. The criteria of flow transition based on an isothermal flow can be misleading. The difference between a heated and an unheated tube flow is by no means small.

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REFERENCES

1. H. Schlichting, *Boundary-layer Theory*, pp. 231–232. McGraw-Hill, New York (1968).
2. B. R. Morton, Laminar convection in uniformly heated horizontal pipes at low Rayleigh numbers, *Q. Jl Mech. appl. Math.* **12**, 410–420 (1959).
3. Y. Mori and K. Futagami, Forced convective heat transfer in uniformly heated horizontal tubes, *Int. J. Heat Mass Transfer* **10**, 1801–1813 (1967).
4. L. S. Yao, Entry flow in a heated straight tube, *J. Fluid Mech.* **88**, 465–483 (1978).
5. L. S. Yao, Free-forced convection in the entry region of a heated straight pipe, *J. Heat Transfer* **100**, 212–219 (1978).
6. S. W. Hong, Theoretical solutions for combined forces and free convection in horizontal tubes with temperature dependent viscosity, *J. Heat Transfer* **98**, 459–465 (1976).
7. J. W. Ou and K. C. Cheng, Natural convection effects on Graetz problem in horizontal isothermal tubes, *Int. J. Heat Mass Transfer* **20**, 953–960 (1977).
8. D. P. Siegwarth, R. D. Mikesell, T. C. Readal and T. J. Hanrathy, Effect of secondary flow on the temperature field and primary flow in a heated horizontal pipe, *Int. J. Heat Mass Transfer* **12**, 1535–1551 (1969).
9. C. A. Hieber and S. K. Sreenivasan, Mixed convection in an isothermal heated pipe, *Int. J. Heat Mass Transfer* **17**, 1337–1348 (1974).
10. B. Zeldin and F. W. Schmidt, Developing flow with combined forced-free convection in an isothermal vertical tube, *J. Heat Transfer* **94**, 211–223 (1972).
11. G. F. Scheele and S. J. Hanratty, Effect of natural convection on stability of flow in a vertical pipe, *J. Fluid Mech.* **14**, 244–256 (1962).
12. G. A. Kemeny and E. V. Somers, Combined free and forced-convection flow in vertical circular tubes, *J. Heat Transfer* **84**, 339–346 (1962).
13. A. Steiner, On the reverse transition of a turbulent flow under the action of buoyancy forces, *J. Fluid Mech.* **47**, 503–512 (1971).
14. L. S. Yao, Free and forced convection in the entry region of a heated vertical channel, *Int. J. Heat Mass Transfer* **26**, 65–72 (1983).
15. C. L. Pekeris, Stability of the laminar flow through a straight pipe of circular cross-section to infinitesimal disturbances which are symmetrical about the axis of the pipe, *Proc. natn. Acad. Sci. U.S.A.* **34**, 285–295 (1948).
16. G. M. Crocos and J. R. Sellars, On the stability of fully developed flow in a pipe, *J. Fluid Mech.* **5**, 97–112 (1959).
17. R. J. Leite, An experimental investigation of the stability of Poiseuille flow, *J. Fluid Mech.* **5**, 81–96 (1959).
18. K. Spielberg and H. Timan, On three- and two-dimensional disturbances of pipe flow, *J. appl. Mech.* **27**, 381–389 (1960).
19. J. A. Fox, M. Lessen and W. V. Bhat, Experimental investigation of the stability of Hagen–Poiseuille flow, *Physics Fluids* **11**, 1–4 (1968).
20. M. Lessen, S. G. Sadler and T. Y. Liu, Stability of pipe Poiseuille flow, *Physics Fluids* **11**, 1404–1409 (1968).
21. A. Davey and P. G. Drazin, The stability of Poiseuille flow in a pipe, *J. Fluid Mech.* **36**, 209–218 (1969).
22. D. M. Burridge and P. G. Drazin, Comments on stability of pipe Poiseuille flow, *Physics Fluids* **12**, 264–265 (1969).
23. H. Salwen and C. E. Grosch, The stability of Poiseuille flow in a pipe of circular cross-section, *J. Fluid Mech.* **54**, 93–112 (1972).
24. V. K. Garg and W. T. Rouleau, Linear spatial stability of pipe Poiseuille flow, *J. Fluid Mech.* **54**, 113–127 (1972).
25. J. T. Sturt, On the non-linear mechanics of wave disturbances in stable and unstable parallel flows, *J. Fluid Mech.* **9**, 353–370 (1960).
26. J. Watson, On the non-linear mechanics of wave disturbances in stable and unstable parallel flows, *J. Fluid Mech.* **9**, 371–389 (1960).
27. W. C. Reynolds and M. C. Potter, Finite-amplitude instability of parallel shear flows, *J. Fluid Mech.* **27**, 465–492 (1967).
28. A. Davey and H. P. F. Nguyen, Finite-amplitude of pipe flow, *J. Fluid Mech.* **45**, 701–729 (1971).
29. N. Itoh, Non-linear stability of parallel flows with subcritical Reynolds number, *J. Fluid Mech.* **82**, 469–479 (1977).
30. H. Zhou, On the non-linear theory of stability of plane Poiseuille flow in the subcritical range, *Proc. R. Soc. London* **A381**, 407–418 (1982).
31. P. K. Sen, D. Venkateswarlu and S. Maji, On the stability of pipe-Poiseuille flow to finite-amplitude axisymmetric and non-axisymmetric disturbances, *J. Fluid Mech.* **158**, 289–316 (1985).
32. T. H. Hughes and W. H. Reid, The stability of spiral flow between rotating cylinders, *Phil. Trans. R. Soc.* **A352**, 351–380 (1968).
33. D. I. Takeuchi and D. F. Jankowski, A numerical and experimental investigation of the stability of spiral Poiseuille flow, *J. Fluid Mech.* **102**, 101–126 (1981).
34. B. S. Ng and E. R. Turner, On the linear stability of spiral flow between rotating cylinders, *Proc. R. Soc. London* **A382**, 83–102 (1982).
35. F. T. Smith and R. J. Dodonyi, On the stability of the developing flow in a channel or circular pipe, *Q. Jl Mech. appl. Math.* **33**, 293–320 (1980).
36. T. J. Hanratty, E. M. Rosen and R. L. Kabel, Effect of heat transfer on flow field at low Reynolds numbers in vertical tubes, *Ind. Engng Chem.* **50**, 815–820 (1958).
37. G. K. Batchelor and A. E. Gill, Analysis of the stability of axisymmetric jets, *J. Fluid Mech.* **14**, 529–551 (1962).
38. D. Gottlieb, M. Y. Hussaini and S. A. Orszag, Theory and application of spectral methods. In *Spectral Methods for Partial Differential Equations* (Edited by Voight, Gottlieb and Hussaini), pp. 1–54. SIAM (1984).
39. C. B. Moler and G. W. Stewart, An algorithm for generalized matrix eigenvalue problems, *SIAM J. Numer. Analysis* **10**, 241–256 (1973).
40. K. S. Gage and W. H. Reid, The stability of thermally stratified plane Poiseuille flow, *J. Fluid Mech.* **33**, (1968).

UN ECOULEMENT NON ISOTHERME PLEINEMENT ETABLI EST-IL POSSIBLE DANS UN TUBE VERTICAL?

Résumé—On présente les résultats d'une analyse de stabilité linéaire de l'écoulement pleinement établi dans un tube chauffé vertical. Ils confirment les observations expérimentales que l'écoulement dans un tube vertical chauffé est instable de façon supercritique. Le nouvel écoulement laminaire stable bifurqué est semblable à celui d'une double spirale. Le mélange induit par ce mouvement peut provoquer un accroissement sensible du transfert de chaleur et, comme il a été observé expérimentalement, retarder la transition à la turbulence.

IST EINE VOLLAUSGEBILDETE NICHTISOTHERME STRÖMUNG IN EINEM SENKRECHTEN ROHR MÖGLICH?

Zusammenfassung—Die Ergebnisse einer linearen Stabilitätsanalyse einer vollausgebildeten Strömung in einem beheizten vertikalen Rohr werden vorgestellt. Sie untermauern die experimentellen Beobachtungen, daß die Strömung in einem beheizten vertikalen Rohr überkritisch instabil ist. Die gespaltene neue, sich im Gleichgewicht befindende, laminare Strömung ist wahrscheinlich eine zweifache Spiralströmung. Durch diese Spiralströmung verursachte Vermischungen können ein beträchtliches Anwachsen des übertragenen Wärmestroms und sogar einen verzögerten Übergang zur Turbulenz hervorrufen, wie experimentell beobachtet wurde.

ВОЗМОЖНО ЛИ ПОЛНОСТЬЮ РАЗВИТОЕ И НЕИЗОТЕРМИЧЕСКОЕ ТЕЧЕНИЕ В ВЕРТИКАЛЬНОЙ ТРУБЕ?

Аннотация—Приведены результаты анализа в линейном приближении устойчивости полностью развитого течения в вертикальной нагреваемой трубе. Они подтверждают тот экспериментальный факт, что течение в нагретой вертикальной трубе является сверхкритически неустойчивым. Новое устойчивое ламинарное течение, вероятно, является двойным спиральным течением. Смешение, индуцированное этим спиральным течением, может вызвать существенное увеличение интенсивности теплообмена и даже, как наблюдалось в экспериментах, задержку перехода к турбулентному режиму.